Quantum Time

John Ashmead

4th Feynman Festival, Olomouc 2009

Quantize time using rules for space, …see what breaks.

www.timeandquantummechanics.com *john.ashmead@timeandquantummechanics.com*

In the world about us, the past is distinctly different from the future. More precisely, we say that the processes going on in the world about us are asymmetric in time, or display an arrow of time. Yet, this manifest fact of our experience is particularly difficult to explain in terms of the fundamental laws of physics. Newton's laws, quantum mechanics, electromagnetism, Einstein's theory of gravity, etc., make no distinction between the past and future - they are time-symmetric.
-- Physical Origins of Time Asymmetry
Halliwell, Pérez-Mercader, & Zurek

This is often the way it is in physics - our mistake is not that we take our theories too seriously,but that we do not take them seriously enough. It is always hard to realize that these numbers and equations we play with at our desks have something to do with the real world. Even worse, there often seems to be a general agreement that certain phenomena are just not fit subjects for respectable theoretical and experimental effort.

-- Steven Weinberg

"The rules of quantum mechanics and special relativity are so strict and powerful that it's very hard to build theories that obey both." -- Frank Wilczek

In relativity time is an equal player with space: apart from the sign of the metric, it is difficult to tell them apart. On the way into a black hole they even appear to change places, formally at least. And to construct a time parameter that works like a normal time parameter in Newtonian physics, reliably going forward more or less uniformly across the universe, is non-trivial.

But in quantum mechanics: time is a parameter, not an operator. It functions like the butler at a party, ushering the players into the room but not taking part in the action itself.

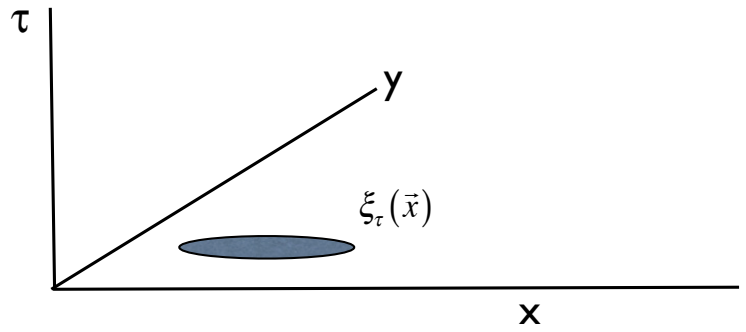
As relativity and quantum mechanics are arguably the two best confirmed theories we have the dichotomy is troubling.

The obvious lines of attack are to add something (another time dimension perhaps), subtract something (find hidden variables that replace quantum theory) or push harder and see what breaks. The first two are perhaps the most popular, we here try the third. We will quantize time using the same rules as we use for space and see what breaks.

To have a reasonable chance at reaching at definite conclusion we will impose limits on the current investigation: we will only look at single particle case, only work to the first approximation, and take the simplest approach available.

Our guiding principle: the most complete symmetry between time and space.

Laboratory time



What Alice measures using wall clocks, laser beams, and grad students

Muga, Mayato, & Egusquiza, Time in Quantum Mechanics, 2001

Hilgevoord & others have emphasized the need to be as precise as practicable in our definition of time.

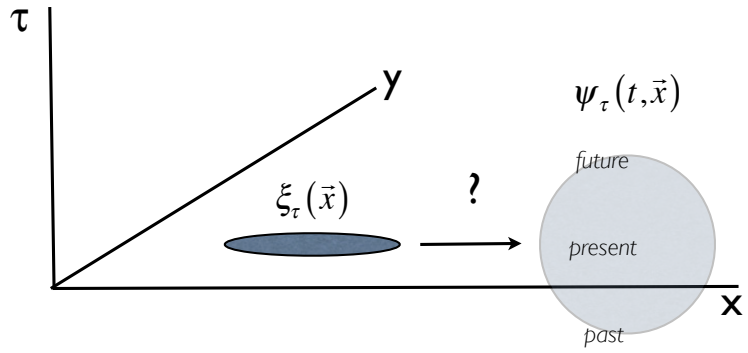
There are 69 definitions of time in the 2nd edition of the Oxford English Dictionary. Once the 3rd edition reaches the 'T's, the situation undoubtedly will be much worse.

The term "laboratory time" has been suggested by Paul Busch, among others. By laboratory time, τ, is defined operationally, as what clocks measure.

We will need to refine the term below; for now, note it is Alice's time, not the particle's proper time. (If we used a particle's proper time we would have difficulty generalizing the work to the multi-particle case; while we are not doing that here, we wish to leave that door open.)

We write the wave function ξ at a particular instant in laboratory time as a function of laboratory time τ, as well as of the usual space coordinates.

Quantum time



``Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality.''
-- MInkowski

The usual wave function is "flat" in time: it represents a well-defined measure of our uncertainty about the particle's position in space, but shows no evidence of any uncertainty in time. This seems very "unquantum-mechanical". Given that any observer, Bob say, going at high velocity with respect to Alice will mix time and space, what to Alice looks like uncertainty only in space to Bob will look like uncertainty in a blend of time and space.

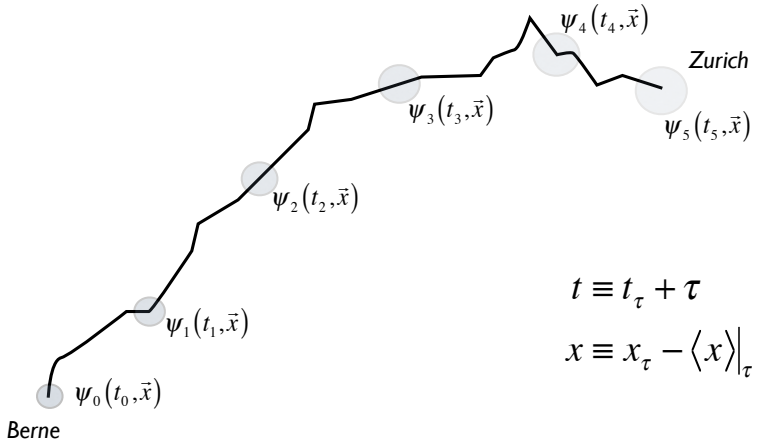
We will therefore extrude Alice's wave function into the time dimension, positing that ψ, at any given instant in laboratory time, is a function of time as well, so that Alice will now be uncertain as to the particle's position in time as well as in space.

We expect the average value of the quantum time at a specific laboratory time will be approximately equal to the laboratory time, and that the average uncertainty in quantum time will normally be small.

It is as if Alice is walking her dog: the dog is usually a bit ahead or behind Alice, but the average position of the dog is same as Alice's and the dog never gets that far from Alice.

Certainly if our analysis shows either these not true, we would have to be very skeptical of "quantum time".

Relative and absolute quantum time



The usual wave function changes shape as laboratory time advances; if it didn't it would not be very interesting. So the quantum time part of the wave function must evolve as well. At each instant in laboratory time, we expect that ψ will have a slightly different shape, if still centered on the current laboratory time.

So it is convenient to distinguish between the absolute quantum time and the relative quantum time. Relative quantum time is defined as the offset in quantum time from the current value of Alice's laboratory time. If the lab clock says 10 seconds past the hour, the relative quantum time might be 10 attoseconds before or after that.

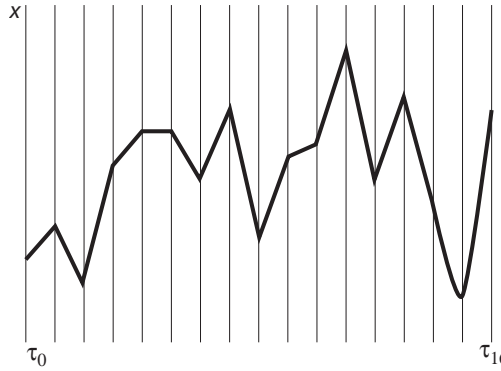
The situation is analogous to the use of "center of mass" coordinates. We use center of mass coordinates to subtract off the average value of the space coordinates, letting us focus on the interesting part. And we can use "center of time" coordinates the same way, to focus on what is essential.

As an example, suppose Alice and Bob are travelling by train from Berne to Zurich. They decide to while away the time by doing quantum mechanical experiments. If they are doing a standard double slit experiment, then they will compute x and y and z relative to their current location on the train. But an outside observer, say Eve, mysteriously left behind at the Berne station, will see the space coordinates in the experiment as the sum of the coordinates Alice and Bob are using plus the space coordinates of the train.

The same with time.

But how are we to actually compute what we expect the wave function at the next instant in laboratory time when we know it at the current instant? We need dynamics.

Path integrals



$$K_T\left(x''_\mu; x'_\nu\right) = \int \mathcal{D}x \exp\left(-i\int_0^T \mathrm{d}\tau L\left[x_\mu, \frac{dx_\mu}{d\tau}\right]\mathrm{d}\tau\right)$$

$$\mathcal{D}x \sim \prod_{n=1}^{n=N} \mathrm{d}^4 x_n$$

$$\psi_T\left(t'', \vec{x}''\right) = \int \mathrm{d}t'\mathrm{d}\vec{x}' K_T\left(t'', \vec{x}''; t', \vec{x}'\right)\psi_0\left(t', \vec{x}'\right)$$

$$\varepsilon \equiv \frac{T}{N}$$

Feynman & Hibbs, Quantum Mechanics and Path Integrals, 1965

Usually we would start an analysis with the Schrödinger equation and then derive the kernel as its inverse. But here it seems more natural to start with the path integral expression for the kernel, and then derive the Schrödinger equation from the kernel. It is a bit like the game Jeopardy, where we are given the answer, the kernel, & have to work out the question, the Schrödinger equation.

Path integrals require only a few simple ingredients: initial wave functions, paths, a Lagrangian, a measure, and a way to add them up.

The simplest paths are a series of delta functions in coordinates; more generally we can take a path as being as a series of wave functions at successive instants, if we have a good basis to work with.

We are using the usual procedure of slicing the laboratory time from start to finish into successively thinner slices, as a frugal delicatessan owner might, and then taking the limit as N, the number of slices, goes to infinity.

Fortunately our rule of treating (quantum) time and space as symmetrically as possible makes most of our decisions trivial. It is largely a matter of replacing three dimensional integrals over the space coordinates with four dimensional integrals over time and the space coordinates. We will have to think a bit about the selection of the Lagrangian and also about convergence and normalization issues.

Lagrangian

$$L\left(t,\vec{x},\frac{dt}{d\tau},\frac{d\vec{x}}{d\tau}\right)=-\frac{m}{2}\left(\frac{dt}{d\tau}\right)^2+\frac{m}{2}\left(\frac{d\vec{x}}{d\tau}\right)^2-e\frac{dt}{d\tau}\Phi(t,\vec{x})+e\frac{d\vec{x}}{d\tau}\cdot\vec{A}(t,\vec{x})-\frac{m}{2}\varepsilon$$

$$\frac{d}{d\tau}\frac{\delta L}{\delta\frac{dx_\mu}{d\tau}}-\frac{\delta L}{\delta x^\mu}=0 \longrightarrow \begin{aligned} m\frac{d^2t}{d\tau^2}&=e\vec{E}\cdot\frac{d\vec{x}}{d\tau}\\ m\frac{d^2\vec{x}}{d\tau^2}&=e\frac{dt}{d\tau}\vec{E}+e\frac{d\vec{x}}{d\tau}\times\vec{B}\end{aligned}$$

$$\bar{L}\left(\vec{x},\frac{d\vec{x}}{d\tau}\right)=\frac{m}{2}\left(\frac{d\vec{x}}{d\tau}\right)^2-e\Phi(\tau,\vec{x})+e\frac{d\vec{x}}{d\tau}\cdot\vec{A}(\tau,\vec{x})$$

$$\frac{d}{d\tau}\left(\left(\frac{dt}{d\tau}\right)^2-\left(\frac{d\vec{x}}{d\tau}\right)^2\right)=0 \longleftarrow \text{conserved quantity}$$

H. Goldstein, Classical Mechanics, 1980

We need to use a Lagrangian which is a Lorentz scalar, completely symmetric between time and space. Consulting our Goldstein, we find such an object. In Goldstein gives a Lagrangian quadratic in the four velocity with respect to an arbitrary scalar parameter, we specialize the arbitrary scalar parameter to laboratory time. The equations of motion are unaffected by this. (Later we will discuss the definition of laboratory time in more detail.)

From the given scalar Lagrangian, we can easily show that the Euler-Lagrange equations give the usual equations of motion for a single charged particle.

The equivalent three dimensional Lagrangian looks a bit different, but gives the same equations of motion, thanks to the fact that the square of the four velocity is a conserved quantity.

The requirements of producing the correct equations of motion, being quadratic in the four-velocity, and giving the correct non-relativistic limit fix our Lagrangian up to a constant. We have chosen the constant, minus one half of the mass, so that our Schrödinger equation will reduce to the Klein-Gordon equation in the appropriate limit, see below.

Discrete Lagrangian

$$L_j \equiv -\frac{m}{2}\left(\frac{x_j - x_{j-1}}{\varepsilon}\right)^2 - e\frac{x_j - x_{j-1}}{\varepsilon}\left(\frac{A(x_j) + A(x_{j-1})}{2}\right) - \frac{m}{2}$$

$$e\frac{d\vec{x}}{d\tau}\cdot\vec{A}(x) \to e\left(\vec{x}_j - \vec{x}_{j-1}\right)\cdot\frac{\vec{A}(x_j) + \vec{A}(x_{j-1})}{2}$$

mid-point rule

$$e\varepsilon\Phi(x_j) \to e\left(t_j - t_{j-1}\right)\frac{\Phi(x_j) + \Phi(x_{j-1})}{2}$$

L. S. Schulman, Techniques and Applications of Path Integrals, 1981

Since we are deriving the kernel by a time-slicing procedure we have to replace the continuous Lagrangian by a discrete one. As is well-known to connoisseurs of path integrals, the process of going from the continuous to the discrete Lagrangian is actually a bit tricky.

In particular, when working with the vector potential we have to use the mid-point rule: replace the vector potential by its average between the before and after slices. By our principle of the most complete symmetry between time and space, since the electric potential is the time component of the four-vector potential, we have to apply the same rule to it.

Convergence

$$K_\tau(x'';x') = C^{N+1}\int \mathrm{d}x_1\mathrm{d}x_2\ldots\mathrm{d}x_N \exp\left(i\varepsilon\sum_{j=1}^{j=N+1} L_j\right)$$

$$-i\frac{m}{2}\left(\frac{t_j - t_{j-1}}{\varepsilon}\right)^2\varepsilon + i\frac{m}{2}\left(\frac{x_j - x_{j-1}}{\varepsilon}\right)^2\varepsilon \qquad \varepsilon \equiv \frac{T}{N}$$

$$\varepsilon \to \varepsilon - i\delta\,?$$

$$m \to m + i\delta\,?$$

H. Kleinert, Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets, 2004
P. Zinn-Justin, Path Integrals in Quantum Mechanics, 2005

We have now lined up an infinite series of integrations. In the usual treatments of path integrals, these do not really converge but are coaxed into doing so by trickery.

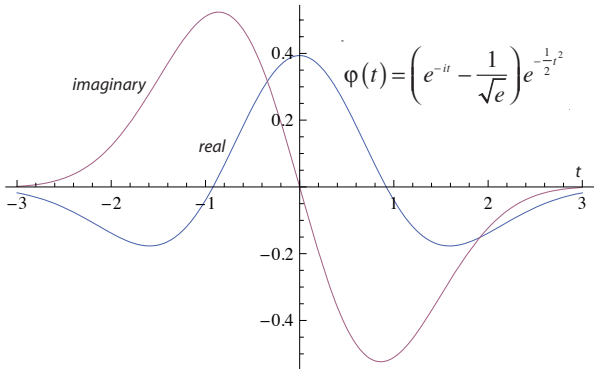
The simplest trick is to add a small imaginary part either to the infinitesimal time ε or to the mass. The problem is that no matter how we do this, either the space integrals diverge or the time integral does. If we use a different sign for the small imaginary part, then we are abandoning our guiding principle of maximum symmetry between time and space.

An alternative procedure, Wick-rotating the time into the complex plane, has the same problem.

What is to be done?

We have not yet given much attention to the initial wave functions or the paths they travel. The usual treatments of path integrals use delta functions and/or plane waves. But these are unphysical.

Morlet wavelets



$$\varphi_{sd}(t) \equiv \frac{1}{\sqrt{|s|}} \left(e^{-if\left(\frac{t-d}{s}\right)} - e^{-\frac{f^2}{2}} \right) e^{-\frac{1}{2}\left(\frac{t-d}{s}\right)^2}$$

G. Kaiser, A friendly guide to wavelets, 1994

We will use Morlet wavelet decomposition to describe our paths.

Wavelets in general are built up by selecting a "mother" wavelet and then scaling and displacing its argument to create a family of wavelets. As in life, so with wavelets: correct choice of your mother is all important.

Morlet wavelets use a sum of two Gaussians as their starting point. If the argument is t or time, then the family of wavelets is produced by displacing the time by d and rescaling it by s. As a result, the Morlet wavelets are Gaussians so play nicely with path integrals.

And, like wavelets in general, Morlet wavelets are well-suited for describing phenomena which are localized in both time and frequency.

Morlet wavelet analysis

$$\psi(t) = \frac{1}{C_f} \int_{-\infty}^{\infty} \frac{\mathrm{d}s\mathrm{d}d}{s^2} \varphi_{sd}(t) \tilde{\psi}_{sd}$$

$$\tilde{\psi}_{sd} = \int_{-\infty}^{\infty} \mathrm{d}t \varphi^*_{sd}(t) \psi(t)$$

$$C_f \equiv 2\pi \int_{-\infty}^{\infty} \frac{\mathrm{d}w}{|w|} \left|\hat{\varphi}(w)\right|^2$$

Any normalizable function may be broken up into a sum over Morlet wavelets. To recreate the original wave function we may write it as a sum over all possible values of the scaling and displacement variables, with appropriate coefficients. The coefficients are given by integrating the complex conjugate of the specific wavelet against the wave function.

The coefficient C needed to recreate the original wave function is given by an integral over the square of the Fourier transform of the mother wavelet. The second Gaussian term has to be present to insure this integral is finite.

The analogy to Fourier analysis is very strong and in fact lurking inside the Morlet wavelet transform is the Fourier transform: if you expand out the original wave function, combining the s, d, and t integrals, you find you are really looking at a thinly disguised version of the Fourier transform, with the coefficient C being just what is needed to "make it all come out".

To handle the four dimensional case we can use products of four Morlet wavelets, one for each dimension, again just as with the Fourier transform.

Now convergence is trivial: the Morlet wavelets are basically Gaussians and enjoy nothing more than converging in the face of path integration. Each wavelet component will converge at its own rate -- this is not uniform convergence -- but we are willing to take convergence by whatever means works. We have shifted the responsibility for convergence from the kernel to the wave function.

Note that this benefit is not specific to the work at hand, it might be useful for ensuing convergence of path integrals in general, not just in the case of temporal quantization.

Normalization

$$1 = \int \mathrm{d}x \left|\varphi_\tau(x)\right|^2$$

$$K_\tau^{(0)}(x'';x') = C_\tau K_\tau^{(raw)}(x'';x')$$

$$C_\tau = \frac{1}{\sqrt{\int \mathrm{d}x'' \left|\int \mathrm{d}x' K_\tau(x'';x')\varphi_0(x')\right|^2}}$$

$$C_\tau = \left(-\frac{im^2}{4\pi^2\varepsilon^2}\right)^{N+1}$$

$$\mathcal{D}x \equiv \left(-\frac{im^2}{4\pi^2\varepsilon^2}\right)^{N+1} \prod_{n=1}^{n=N} \mathrm{d}^4 x_n$$

$$K_\tau(x'';x') = \int \mathcal{D}x \exp\left(-i\sum_{j=1}^{N+1} m\frac{\left(x_j - x_{j-1}\right)^2}{2\varepsilon} - ie\left(x_j - x_{j-1}\right)\frac{A\left(x_j\right) + A\left(x_{j-1}\right)}{2} - i\frac{m}{2}\tau\right)$$

Our integrals now converge; but are they normalizable?

Given that the wave function is normalized at start, does it stay that way?

We first look at a single wavelet, more precisely at a single Gaussian function.

We write the normalized kernel as a normalization coefficient times the "raw" kernel.

We deal here only with the free case; once we have derived the Schrödinger equation we will use that to establish unitarity in general.

A straightforward calculation gives the normalization constant. While the calculation was done for an arbitrary Gaussian, it shows the normalization is independent of the specifics of the Gaussian, i.e. its scale and displacement to use wavelet terminology.

If this were not true, then a dark cloud of suspicion would hang over the entire project. The usual path integrals are normalized in a way that is independent of the wave function; ours should be as well.

With the normalization in hand, we can write out the full expression for the path integral kernel for a charged particle in an electro-magnetic field.

Schrödinger equation

$$\psi_{\tau+\Delta\tau}(x) = \int \mathrm{d}^4 x K_\varepsilon(x;x')\psi_\tau(x')$$

$$i\frac{d\psi_\tau}{d\tau} = \frac{1}{2m}\partial^\mu\partial_\mu\psi_\tau + \frac{ie}{m}\left(A^\mu\partial_\mu\right)\psi_\tau + \frac{ie}{2m}\psi_\tau\left(\partial^\mu A_\mu\right) - \frac{e^2}{2m}A^\mu A_\mu\psi_\tau + \frac{m}{2}\psi_\tau$$

$$i\frac{\partial}{\partial t} \to E$$
$$-i\vec{\nabla} \to \vec{p}$$

$$\longrightarrow \quad i\frac{d\psi_\tau(x)}{d\tau} = \left(-\frac{\left(E - e\Phi(x)\right)^2}{2m} + \frac{\left(\vec{p} - e\vec{A}(x)\right)^2}{2m} + \frac{m}{2}\right)\psi_\tau(x)$$

Now we derive the Schrödinger equation from the kernel.

We look at the infinitesimal kernel, the kernel to go one time-slice forwards. We can use this to write the wave function at the next slice in terms of the wave function and its derivatives at the previous time slice. At the end we are taking the limit as ε goes to zero, so can discard terms higher order in ε.

This is the reverse of the procedure used, as in Schulman, to derive the path integral expression from the Schrödinger equation. It is a bit like turning an engine into a refrigerator by running it the wrong way. (The reason we had to worry so much about the normalization is that that is something which would normally be provided implicitly by the Schrödinger equation.)

Unitarity

$$\dot{\psi} = \frac{-i}{2m}\partial^{\mu}\partial_{\mu}\psi + \frac{e}{m}(A\partial)\psi + \frac{e}{2m}(\partial A)\psi + i\frac{e^2}{2m}A^{\mu}A_{\mu}\psi - i\frac{m}{2}\psi$$

$$\dot{\psi}^{*} = \frac{i}{2m}\partial^{\mu}\partial_{\mu}\psi^{*} + \frac{e}{m}(A\partial)\psi^{*} + \frac{e}{2m}(\partial A)\psi^{*} - i\frac{e^2}{2m}A^{\mu}A_{\mu}\psi^{*} + i\frac{m}{2}\psi^{*}$$

$$P \equiv \int \mathrm{d}^4 x \psi^{*}(x)\psi(x)$$

$$\dot{P} = \int \mathrm{d}^4 x \left(\psi^{*}(x)\dot{\psi}(x) + \dot{\psi}^{*}(x)\psi(x)\right)$$

$$\dot{P} = 0$$

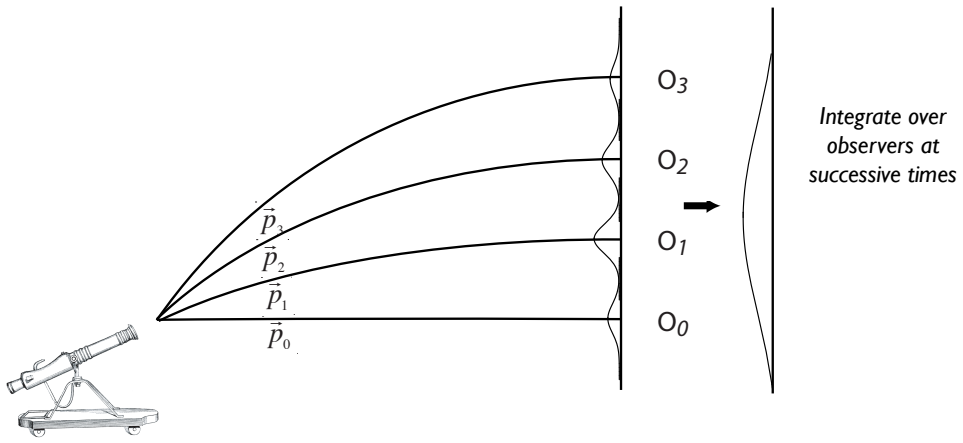
To show unitarity in the general case, we use the complex conjugate of the Schrödinger equation to get the time derivative (with respect to laboratory time) of the complex conjugate of the wave function.

Then we define P as the norm of the wave function at a particular time.

We take the derivative of P with respect to the laboratory time.

And by applying the Schrödinger equation and doing some integrations by parts in the usual way we can show that the derivative of the norm with respect to laboratory time is zero.

Geodesic time



We have still one more problem. While most of the elements of our path integrals are invariant under Lorentz transformations, the laboratory time itself is not.

If Alice is at rest in her laboratory, while Bob is jetting around like a fusion powered mosquito, they will have very different definitions of the laboratory time. This cannot be. We need to define the laboratory time in a way that is manifestly independent of the observer.

Consider the case of a particle starting at a particular time but then being detected by a detector that is moving in time (i.e. just sitting there on the lab bench). Let us consider the initial wave function as composed of a set of components with different values of the three momentum (or more accurately four momentum). Each component has a natural geodesic that will get it to the detector at a slightly different time. We break up the wave function into these components and use as the laboratory time for each component the geodesic time to the detector. The geodesic time is an invariant, so -- at the cost of a bit of algebra -- we have an observer independent laboratory time.

Of course, at the end of the maneuver, we have to put the pieces of the wave function back together again. But such is the price of symmetry.

Semi-classical approximation

$$K_\tau(x'';x') = \int \mathcal{D}x \exp\left(-i\sum_{j=1}^{N+1} m\frac{\left(x_j - x_{j-1}\right)^2}{2\varepsilon} - ie\left(x_j - x_{j-1}\right)\frac{A\left(x_j\right)+A\left(x_{j-1}\right)}{2} - i\frac{m}{2}\tau \right)$$

$$\frac{\delta}{\delta x_j} i\varepsilon \sum_{j=1}^{j=N+1}\left(-\frac{m}{2}\left(\frac{x_j - x_{j-1}}{\varepsilon}\right)^2 - e\frac{x_j - x_{j-1}}{\varepsilon}\left(\frac{A\left(x_j\right)+A\left(x_{j-1}\right)}{2}\right) - \frac{m}{2}\right) = 0 \rightarrow x_j = \bar{x}_j$$

$$\int \mathcal{D}x \exp\left(-i\varepsilon \sum_{j=1}^{j=N+1} \frac{1}{2}\frac{\partial^2 L\left(\bar{x}, \frac{d\bar{x}^\mu}{d\tau}\right)}{\partial\delta x^\mu \partial\delta x^\nu}\delta x^\mu \delta x^\nu + O(\delta x)^3 \right) = \sqrt{\det\left(-\frac{\partial^2 S_\tau(\bar{x}'';\bar{x}')}{\partial x' \partial x''}\right)}$$

$$K_\tau(x'';x') \approx \frac{1}{\sqrt{2\pi i}^4}\sqrt{\det\left(-\frac{\partial^2 S_\tau(\bar{x}'';\bar{x}')}{\partial x' \partial x''}\right)} \exp\left(iS_\tau(\bar{x}'';\bar{x}')\right)$$

It is the outstanding feature of the path integral that the classical action of the system has appeared in a quantum mechanical expression, and it is this feature that is considered central to any extension of the path integral formalism. -- Mark Swanson, Path Integrals and Quantum Processes.

Now that we have our kernel and Schrödinger equation, we have to show we get the usual results in the appropriate limit (i.e. when we are looking at static systems) and solve actual problems.

For cases when the Lagrangian is quadratic in the time and space coordinates, we may solve for the kernel exactly using the semi-classical approximation. When the Lagrangian is not quadratic the semi-classical approximation can still be reasonable.

The basic idea is the same here as in the usual case, basically steepest descents done slice by slice. We expand the Lagrangian in powers of the differences of the coordinates from the average trajectory. We see that the equation for the average is exactly the discretized classical equations of motion, so the average is given by the classical trajectory.

We have a lot of integrals of exponentials of quadratic argument to do, but the analysis goes through in the case of quantum time with essentially no formal change from the analysis for three dimensions. Various integrals are over four rather than three dimensions and convergence is guaranteed by uttering the mantra "Morlet wavelet decomposition" rather than the mantras "small imaginary component to the mass" or "Wick rotation", but the rest of the treatment is unchanged.

The result is formally identical to the result in the three-dimensional case, except that we have one extra factor of one over the square root of 2πi.

This implies that we can see the temporal quantization case as being the standard quantum results plus a bit of fuzz in the time direction. And this in turn helps explain why, if real, quantization in time might not have been seen, sans purpose built experiments. Quantum effects in general aren't that visible if you aren't looking for them; quantum effects in time the same.

Stationary states

$$i\frac{d\psi_\tau(x)}{d\tau} = \left(-\frac{\left(E - e\Phi(x)\right)^2}{2m} + \frac{\left(\vec{p} - e\vec{A}(x)\right)^2}{2m} + \frac{m}{2}\right)\psi_\tau(x)$$

$$\frac{d\psi_\tau(x)}{d\tau} = 0 \Rightarrow \left(-\left(E - e\Phi\right)^2 + \left(\vec{p} - e\vec{A}\right)^2 + m^2\right)\psi_\tau(x) = 0$$

$$\tau_e \equiv \frac{\hbar}{m_e c^2} = \frac{6.58 \cdot 10^{-16}\, eVs}{.511 \cdot 10^6\, eV} = 1.29 \cdot 10^{-21}\, s = 1.29 zs$$

With that said, we would like to look a bit harder at how we go from the four to three dimensions.

If we look at the Schrödinger equation in four dimensions, we see that the stationary solutions, those where the wave function is not a function of laboratory time, are given by the solutions of the Klein-Gordon equation.

Is this a reasonable condition?

We expect that in the free case, characteristic times will be of the order of the Compton time of the particle, the lighter the longer. For an electron, this is still of order zepto-seconds.

Unless special steps are taken, wave functions which vary at time scales much longer than this will tend to pick out the stationary (in laboratory time) states.

Schrödinger equation in relative time

$$t_\tau \equiv t - \tau$$

$$i\frac{d}{d\tau}\psi_\tau^{(rel)}(t_\tau,\vec{x}) = i\partial_{t_\tau}\psi_\tau^{(rel)}(t_\tau,\vec{x}) + \left(-\frac{1}{2m}\left(i\frac{\partial}{\partial t} - e\Phi(t,\vec{x})\right)^2 + \frac{\vec{p}^2}{2m} + \frac{m}{2}\right)\psi_\tau^{(rel)}(t_\tau,\vec{x})$$

$$i\frac{d}{d\tau}\bar{\psi}_\tau(\vec{x}) = \left(\frac{\vec{p}^2}{2m} + e\Phi(t,\vec{x})\right)\bar{\psi}_\tau(\vec{x})$$

$$H = \hat{H}(t_\tau) + \bar{H}(\vec{x}) + V_\tau(t_\tau,\vec{x})$$

$$\psi'_\tau(t_\tau,\vec{x}) = \exp\left(ie\Phi(\vec{x})t_\tau\right)\psi_\tau(t_\tau,\vec{x})$$

$$H = \left(-\frac{E^2}{2m} + \frac{m}{2}\right) + \left(\frac{\vec{p}^2}{2m} + e\Phi(\vec{x})\right) + \left(\frac{e\vec{p}\cdot\vec{E}(\vec{x})}{2m}t_\tau + \frac{e^2\vec{E}^2(\vec{x})}{2m}t_\tau^2\right)$$

$$\psi'_\tau(t_\tau,\vec{x}) \approx \sqrt[4]{\frac{1}{\pi\sigma_t^2}}\exp\left(-iEt_\tau - \frac{t_\tau^2}{2\sigma_t^2}\right)\bar{\psi}_\tau(\vec{x}) \longrightarrow t_\tau \to \langle t_\tau\rangle \approx -iE, t_\tau^2 \to \langle t_\tau^2\rangle = \frac{\sigma_t^2}{2}$$

If we work with block time we see the maximum symmetry between time and space. Block time is therefore preferred for defining the rules for quantum time.

But we expect that in general wave functions in time will be centered on the laboratory time, so to see the connection between temporal quantization and the three dimensional approach, it is best to work with the relative time, to subtract off the laboratory or "center of time" part.

We can easily work out the Schrödinger equation using relative time. We here look at the case where there is only an electric potential, first giving the four dimensional and then the three dimensional Schrödinger equation. We can write the full Hamiltonian in a natural way as a time part, a space (standard part), and correction potential.

With the help of a gauge transformation, we get the effective Hamiltonian. The first two parts can be solved by taking sums over direct product wave functions and using separation of variables. Then we calculate the corrections from the correction potential using first order perturbation theory. (If we need to use higher order corrections we are already outside the scope of this investigation.)

For one of our Gaussian test functions, the linear term will usually be zero; the quadratic may be estimated from the dispersion in time.

Bound states

$$H^{(bound)}(t,\vec{x}) = -\frac{1}{2m}\left(E - e\Phi(\vec{x})\right)^2 + \frac{1}{2m}\vec{p}^2 + \frac{m}{2}$$

$$\boxed{H^{(bound)}(t,\vec{x})\psi(t,\vec{x}) = 0}$$

$$\bar{H}(\vec{x}) = \sqrt{m^2 + \vec{p}^2} - m + e\Phi(\vec{x})$$

$$\bar{H}(\vec{x}) \approx \frac{\vec{p}^2}{2m} + e\Phi(\vec{x})$$

$$\boxed{\bar{H}(\vec{x})\xi_{\bar{n}}(\vec{x}) = \bar{\mathcal{E}}_{\bar{n}}\xi_{\bar{n}}(\vec{x})}$$

$$\bar{n} \equiv \{n,l,m\}$$

$$\psi_{E,\bar{n}}(t,\vec{x}) = \chi_E(t)\xi_{\bar{n}}(\vec{x})$$

$$\chi_E(t) = \frac{1}{\sqrt{2\pi}}\exp(-iEt)$$

$$\langle\psi|H|\psi\rangle = 0$$

$$E = m + \bar{\mathcal{E}}_{\bar{n}}$$

$$\boxed{\psi_{E,\bar{n}}(t,\vec{x}) = \frac{1}{\sqrt{2\pi}}\exp\left(-i\left(m + \bar{\mathcal{E}}_{\bar{n}}\right)t\right)\xi_{\bar{n}}(\vec{x})}$$

It takes about 150 attoseconds for an electron to circle the nucleus of an atom. An attosecond is 10 to the minus 18th seconds long, or, expressed in another way, an attosecond is related to a second as a second is related to the age of the universe. -- Mauritsson

Why are there bound states?

In the Bohr picture, we see only those orbits which equal themselves at the ends of an orbit. This makes sense for an electron confined in a box.

But the atom is a box only in three dimensions; in the fourth it is open-ended. A closure condition is much less natural in the case of temporal quantization.

Here we again invoke the stationary state condition; for interaction times much greater than the "orbit time" of an atom, only the stationary states will matter. We therefore use the stationary state condition as the defining condition for bound states, giving a four dimension equation for the atom, as shown.

We compare this to the Hamiltonian (barred) for the three dimensional case.

We can write a test solution, an ansatz, for the four dimensional solution as the product of a plane wave in time with one of the usual three-space solutions. These have zero expectation for the Hamiltonian, if we select the energy of the time part to be the mass plus the usual binding energy of the bound state.

In general we expect real wave functions in time will be a bit more Gaussian than plane, a kind of cloud in time that spins around the more stately three dimensional part. But we expect cloud in time will be centered on the plane wave solution. We may estimate the dispersion of the time cloud as being about the orbit time of the three dimensional state. (Giving a more definitive answer would require that we treat the multi-particle case; that would be another investigation.)

Experimental tests

- Perhaps 300 experiments in Auletta alone
- Interchange time and a space dimension, get a test of quantum time
- We look at a few here

S. K. Lamoreaux, A Review of the Experimental Tests of Quantum Mechanics, 1992

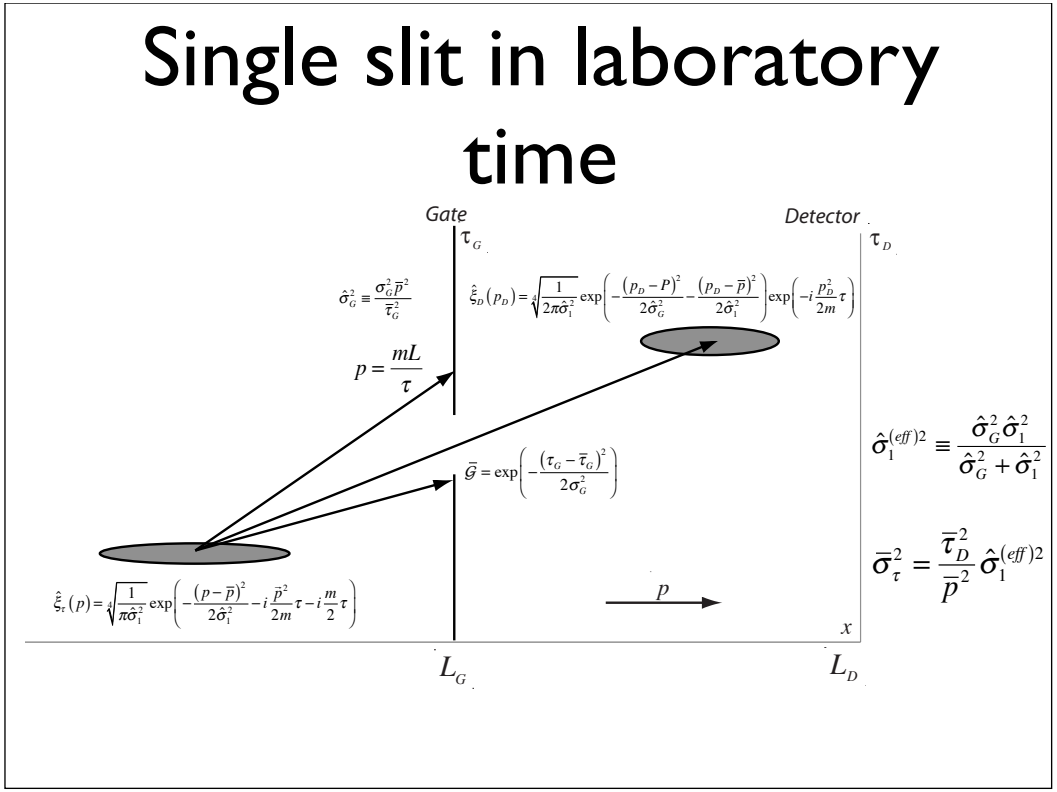
P. Ghose, Testing quantum mechanics on new ground, 1999

G. Auletta, Foundations and Interpretation of Quantum Mechanics: In the Light of a Critical-Historical Analysis of the Problems and of a Synthesis of the Results, 2000

If temporal quantization has any particular merit, it is that it is an experiment factory. Practically any foundational experiment may be turned into a test of temporal quantization by flipping time and a space dimension.

In Auletta's compendium of foundational experiments he lists perhaps three hundred, that is therefore a rough lower bound on the number of possible tests of temporal quantization.

We look at only a few possibilities, just enough to firm up our understanding of quantum time.



We now look at the classic single slit experiment in time.

We'll use the three dimensional wave function as a kind of carrier. More accurately, we will look at the wave function in momentum in the x direction, ignoring the other two.

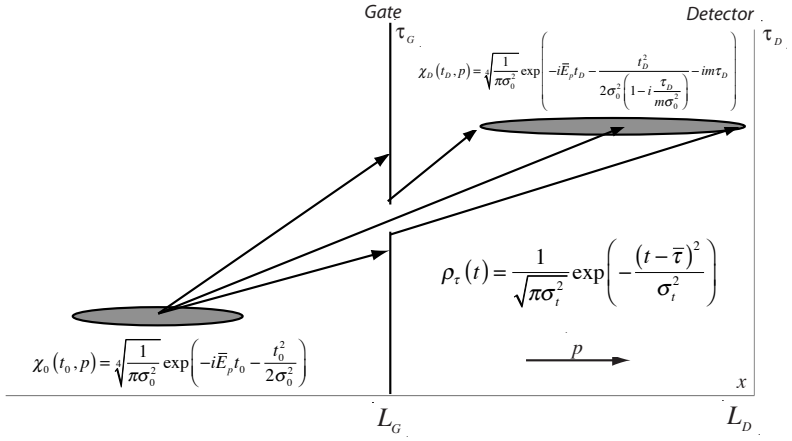
In the simplest case, if the momentum is in the "sweet spot", not too fast not too slow, the particle gets through. If not, not. (Effectively, we are treating momentum and laboratory-time-to-target as inversely related variables.)

We will a Gaussian gate, an idea suggested by Feynman & Hibbs. This produces gates which are a bit more mathematically tractable. And, given our earlier remarks on Morlet wavelets, still fully general.

WIth these assumptions, the calculations are doable. The time gate in standard quantum mechanics can only narrow the beam. If the gate is much narrower than the beam, the resulting width will be the width of the gate. If the beam is much narrow than the gate, the resulting width will be the width of the beam. A full treatment will soften this conclusion, but the essential idea remains.

We give the dispersion in momentum, most directly computed, and also the dispersion in laboratory time.

And in quantum time



$$\sigma_t^2=\frac{\sigma_G^2\sigma_0^2}{\sigma_G^2+\sigma_0^2}+\frac{1}{m^2}\left(\frac{1}{\sigma_0^2}+\frac{1}{\sigma_G^2}\right)\bar{\tau}_D^2+\frac{\hat{\sigma}_1^2}{\bar{p}^2}\bar{\tau}_D^2$$

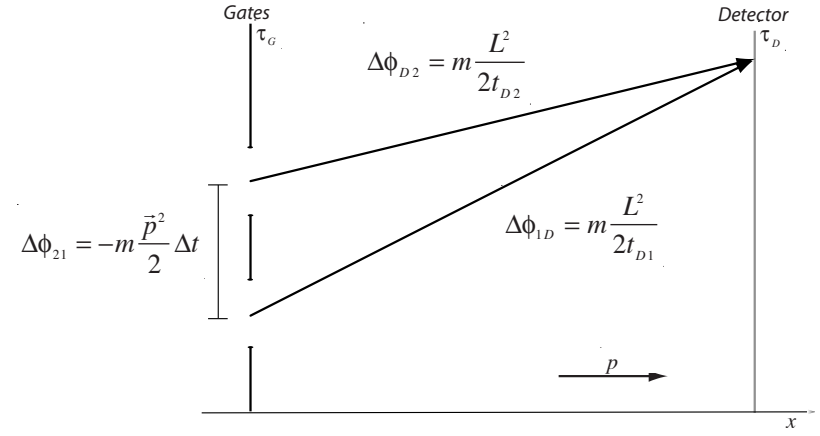
We solve the four dimensional case using the three dimensional result as a carrier. For each ray (specific value of momentum) have an associated wave function in time associated with its wave function in space.

We apply the four dimensional kernel to these wave functions and sum over all rays to get the wave function at the detector.

We find the dispersion post gate, in time, is given by the sum of the dispersion of the time and space parts.

The striking point here is that now the dispersion in time has a term of form one over the dispersion (squared) of the gate. Now we can make the dispersion (in time) of the wave function as great as we wish. This is almost opposite the situation with the standard theory, where we can only make the dispersion smaller. There are significant experimental possibilties here.

Double slit in time

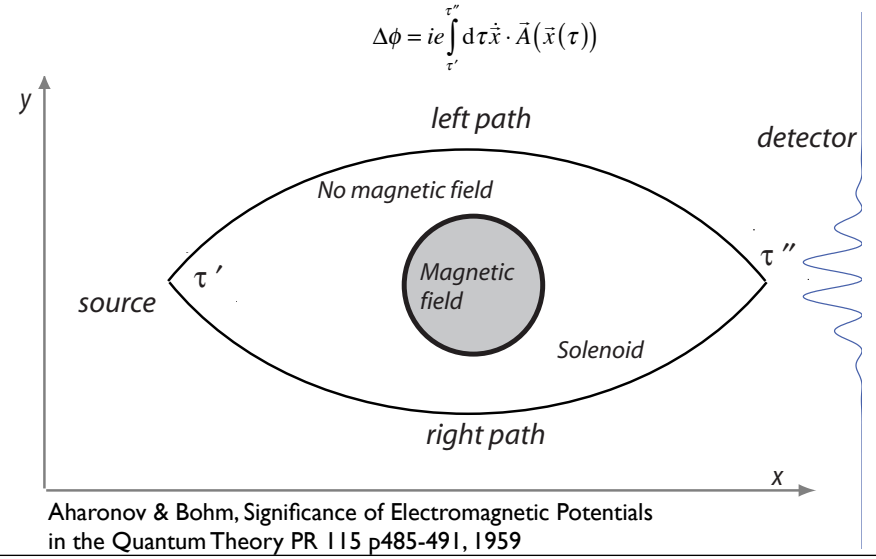


$$\exp(-iEt) = \exp\left(-i\sqrt{m^2+\vec{p}^2}t\right) = \exp\left(-imt - i\frac{\vec{p}^2}{2m}t\right) \xrightarrow{\langle t\rangle \to \tau} \exp\left(i\frac{\vec{p}^2}{2m}\tau\right)$$

Strikingly enough, if we define a standard quantum mechanical double slit in time experiment using plane waves in the usual ad hoc way we see no interference: the total phase to detector associated with the time part is the same for each path: the earlier path accumulates less phase difference before the gate; more after. The other the opposite.

Fortunately the situation with our Gaussian test functions is at least a bit more interesting. If we treat each gate as a source of Gaussian test functions, we will see interference in both the standard and temporal quantization cases. With temporal quantization, the fringe pattern is unaffected. However the width of each fringe is increased exactly as with the single slit experiment.

Aharonov-Bohm experiment



Aharonov & Bohm, Significance of Electromagnetic Potentials in the Quantum Theory PR 115 p485-491, 1959

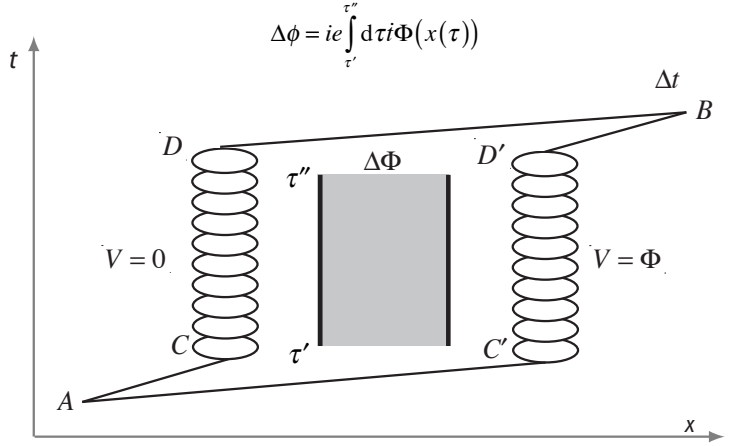
The Aharonov-Bohm effect provides a very striking indication that the vector potential is more fundamental than the magnetic field.

A particle's wave function is split, with the two halves being routed around a solenoid. There is no magnetic field outside the solenoid, but there is vector potential. In semi-classical approximation, the phase shift is given by the integral of the velocity dot the vector potential. The phase shift difference between the two paths depends on the vector potential and can be tuned by changing the field.

Tuning the phase shift difference moves the interference pattern. This effect has been seen. Extraordinary!

Further, Aharonov & Rohrlich, Quantum Theory for the Perplexed, 2005, argue that without this sort of phase interference it would be possible to break the Heisenberg uncertainty principle, so that the effect is fundamental.

Aharonov-Bohm in time



When you interchange time and a space dimension you also interchange the electric and magnetic fields. It is therefore logical to look for an analog to the Aharonov-Bohm effect in time using the electric field.

We here consider a capacitor which can be turned on and off and which has small hole drilled in it so that a charged particle can get through.

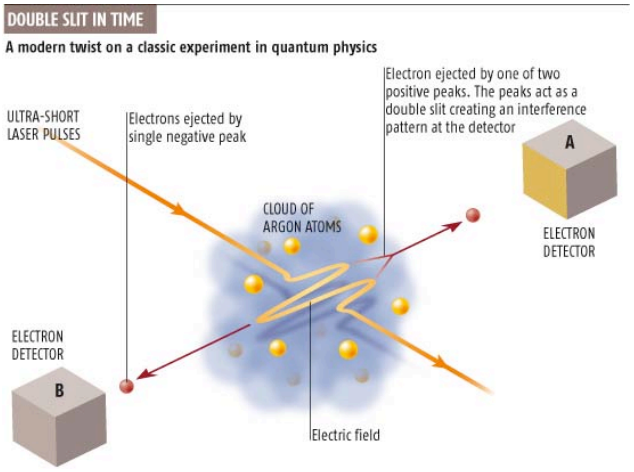
We start with the capacitor off. We split a particle's wave function into two parts. We send the first part through the capacitor and into a delay loop on the far side, a loop which, needless to say, preserves the phase of the particle. Meanwhile the second half is put into a similar delay loop on the near side. Then the capacitor is turned on. After a bit of time, we turn off the capacitor, and send the nearside part through the capacitor to be merged with its other half.

Again in semi-classical approximation, the phase shift associated with each part is given by the integral of the potential times the velocity of quantum time with respect to laboratory time. For non-relativistic particles, this velocity is about one so we then talking about the integral of the potential over the laboratory time. The potential on each side of the capacitor is different, so that the phase shifts each side of the wave function sees are different. The difference is proportional to the potential difference on the capacitor.

As there is a phase shift difference between the before and after parts, we can see an interference pattern and, just as in the case of the Aharonov-Bohm effect, we can control the interference pattern by changing the potential on the capacitor, so that the two halves are affected by a field they never see.

We note that the same effect is expected in the three dimensional theory, though with slightly different estimates of the phase in the relativistic case as the velocity of quantum time with respect to laboratory time is replaced by one. For the Aharonov-Bohm effect in time, the benefit of temporal quantization is not that the effect is new with temporal quantization but rather that we are led to ask questions that might otherwise not occur to us.

Lindner's double slit in time



Lindner et al, "Attosecond double-slit experiment" PRL **95** 040401

Perhaps the closest thing to a direct test of quantization in time in the literature is Lindner's Double Slit in Time experiment.

In this edge of the art experiment, a high energy very fast pulse is sent through cloud of argon atoms. One side of the electric pulse has one peak, the other two. An electron may be ejected from the atom by either the single pulse or, in keeping with the general principles of quantum mechanics (and some very difficult specific calculations), by the two acting in correlation.

If ejected by the one, we have a single slit in time, if by two, a double slit.

Lindner's experiment escapes our strictures against interference in the double slit experiment in three-dimensional quantum mechanics because each peak acts as (correlated) source.

Per the earlier analysis we expect no shift in the fringes but we do expect a widening of the fringes. More specifically, we expect that to the dispersion otherwise expected we will add the dispersion associated with the width of the atomic wave function in time. Antecedent to an analysis of the multi-particle case, we can only give an estimate of that width. But the orbit time of the electron should give an order of magnitude estimate. Anomalous widening of the individual fringes would be an indication of temporal quantization.

Further experiments

- Time-varying magnetic fields
- Constant electric fields
- Time-varying electric fields

Using the relative time form of the Schrödinger equation, we can easily compute first order corrections for the progress of a time-extended wave function through time-varying magnetic fields, and constant and time-varying electric fields.

Note in particular that because the four dimensional Lagrangian couples the derivative of quantum time with respect to laboratory time to the electric potential, we see some effect of temporal quantization even for constant electric fields.

Quantum Time

John Ashmead

$$K_\tau(x'';x') = \int \mathcal{D}x \exp\left(-i\sum_{j=1}^{N+1} m\frac{\left(x_j - x_{j-1}\right)^2}{2\varepsilon} - ie\left(x_j - x_{j-1}\right)\frac{A\left(x_j\right) + A\left(x_{j-1}\right)}{2} - i\frac{m}{2}\tau \right)$$

$$i\frac{d\psi_\tau(x)}{d\tau} = \left(-\frac{\left(E - e\Phi(x)\right)^2}{2m} + \frac{\left(\vec{p} - e\vec{A}(x)\right)^2}{2m} + \frac{m}{2} \right)\psi_\tau(x)$$

www.timeandquantummechanics.com john.ashmead@timeandquantummechanics.com

Wheeler's often unconventional vision of nature was grounded in reality through the principle of radical conservatism, which he acquired from Niels Bohr: Be conservative by sticking to well-established physical principles, but probe them by exposing their most radical conclusion
- Kip S. Thorne

The world cannot be half quantum mechanical, half classical. -- Feynman & Hibbs, Path integrals and quantum mechanics.

The two equivalent results of this investigation are displayed here: the path integral expression for the kernel and the corresponding Schrödinger equation, inverses of each other.

Perhaps the most striking thing about this program of quantizing time is that it does not fail "out-of-hand". Certainly it has had a number of opportunities to do so. We might not have been able to ensure convergence or normalization of the path integrals, we might have not been able to find a suitable Lagrangian, there was no guarantee we would reproduce either the free behavior or the bound state energy level condition, the predicted effects might have been so large they were sure to have already been seen, or on the other hand we might have found ourselves with a conjecture which made no testable predictions.

While I've tried to pick out the most likely approach, there is an unavoidable of admixture of conjecture in any treatment such as this. It is as if we were in a CAD/CAM program and hit the extrude button to project a two dimensional object into the third. While there is a usually a simplest way to do this, there is seldom a unique way.

But given that temporal quantization is something of an experiment factory, all that we really require from this investigation is some reasonable order of magnitude estimates and a few ideas on how to use them. Whether time is flat or fuzzy or something inbetween is then an experimental question.

I would like to particularly thank my old friend Dr. Jonathan M. Smith for his help and guidance in this work, to thank the Feynman Festival in its current & earlier incarnations, to thank QUIST similarly, to thank D. Dugan, L. Kalb & F. C. Welch for encouragement & support, and to thank many physicists including S. Personick, J. Myers, G. Bowson, F. Herz, Y. S. Kim, M. Noz, J. Wile, M. Riesen, & someone I only know as "Terry the Physicist" for much useful criticism. And finally, I would like to raise a toast to the six German students who, over a round of excellent Czech beer at the Cafe Destiny in the walled city of Olomouc, helped to put this work into its final form.